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THE REGIONS OF INSTABILITY OF A SYSTEM WITH A PERIODICALLY VARYING MOMENT OF INERTIA†

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The problem of finding the regions of instability of a system with a periodically varying moment of inertia is considered. An equation is derived which describes small torsional oscillations of a system with periodic coefficients, which depend on four constant parameters, including damping. A method of investigating stability based on an analysis of the behaviour of Floquet multipliers is described. Analytical expressions are obtained for the regions of instability (parametric resonance) in parameter space. Numerical examples are given.

1. FORMULATION OF THE PROBLEM

Consider small torsional oscillations of a vertical elastic rod with a horizontal disc with a flange rigidly attached to it (Fig. 1). The ends of a rigid spoke are connected to the disc, and two symmetrically situated point masses m can slide along this spoke. We will assume that these masses oscillate along the radius of the disc, symmetrically about the axis, as given by the periodic relation

$$r = r_0 + a\varphi(\Omega t), \quad \int_0^{2\pi} \varphi(\tau)d\tau = 0 \tag{1.1}$$

where r_0 is the mean distance from the mass to the elastic axis of the disc, a and Ω are the amplitude and frequency of the excitation, respectively, and $\varphi(\tau)$ is a smooth periodic function of period 2π with zero mean value. The moment of inertia of the system (the disc with the two masses) is then

$$J(t) = J_0 + 2m[r_0 + a\varphi(\Omega t)]^2 \tag{1.2}$$

where J_0 is the moment of inertia of the disc. The equation of small torsional oscillations of the system has the form

$$(J(t)\dot{\theta})^{\cdot} + \gamma\dot{\theta} + c\theta = 0 \tag{1.3}$$

where θ is the angle of torsion, c is the stiffness of the rod and a dot denotes a derivative with respect to time t. The amplitude a and the damping factor γ are assumed to be small.

The problem is to find the values of the parameters for which the trivial equilibrium position $\theta = 0$ becomes unstable. This problem was formulated previously in [1] without damping and with a harmonic excitation.

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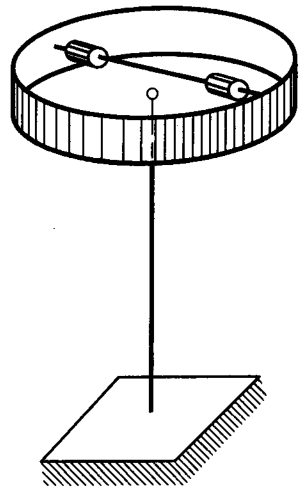

Fig. 1

We will introduce the following dimensionless quantities and parameters

$$\tau = \Omega t, \quad \varepsilon = \frac{a}{r_0}, \quad \varsigma = \frac{2mr_0^2}{\tilde{J}_0}, \quad \tilde{J}_0 = J_0 + 2mr_0^2$$
$$\beta = \frac{\gamma}{\sqrt{\tilde{J}_0 c}}, \quad \omega = \frac{1}{\Omega}\sqrt{\frac{c}{\tilde{J}_0}}, \quad x_1 = \theta, \quad x_2 = \frac{\tilde{J}(t)\dot{\theta}}{\Omega}, \quad \tilde{J}(t) = \frac{J(t)}{\tilde{J}_0} \tag{1.4}$$

The Eq. (1.3) can be written in the form of a system of first-order equations

$$\frac{dx_1}{d\tau} = \frac{1}{\tilde{J}(\tau)}x_2, \quad \frac{dx_2}{d\tau} = -\omega^2 x_1 - \frac{\beta\omega}{\tilde{J}(\tau)}x_2 \tag{1.5}$$

with the relation

$$\tilde{J}(\tau) = 1 + 2\varepsilon\varsigma\varphi(\tau) + \varepsilon^2\varsigma\varphi^2(\tau) \tag{1.6}$$

In the new variables the assumption that the periodic function $\varphi(\tau)$ is smooth can be relaxed and one can merely assume that it is piecewise-continuous.

The right-hand sides of Eqs (1.5) are linear functions of the vector $\mathbf{x} = (x_1, x_2)$, periodic in τ with period 2π. Equations (1.5) and (1.6) depend explicitly on four independent parameters ς, ω, ε and β, of which the last two are small

$$0 < \omega, \quad 0 < \varsigma < 1, \quad 0 \le \varepsilon \ll 1, \quad 0 \le \beta \ll 1 \tag{1.7}$$

The problem is to find the regions of instability (parametric resonance) of the trivial solution $\mathbf{x} = 0$ in the three-dimensional space $\mathbf{p} = (\omega, \varepsilon, \beta)$ for a fixed value of the parameter ς.

2. DERIVATIVES OF THE MONODROMY MATRIX WITH RESPECT TO THE PARAMETERS

Consider the system of linear equations

$$\dot{\mathbf{x}} = \mathbf{G}\mathbf{x} \tag{2.1}$$

where $\mathbf{G} = \mathbf{G}(t, \mathbf{p})$ is an $N \times N$ square matrix, which depends smoothly on the vector of real parameters $\mathbf{p} = (p_1, p_2, \ldots p_n)$ and is a continuous periodic function of time with period T.

The fundamental matrix $\mathbf{X}(t)$ of system (2.1) is found from the matrix differential equation with initial condition

$$\dot{\mathbf{X}} = \mathbf{G}\mathbf{X}, \quad \mathbf{X}(0) = \mathbf{I} \tag{2.2}$$

where $\mathbf{I}$ is the identity matrix and is called the matriciant. The monodromy (Floquet) matrix is defined by the equality $\mathbf{F} = \mathbf{X}(T)$ [2, 3].

To investigate the stability of linear system (2.1) we use Floquet's theory, according to which a linear system with periodic coefficients is stable if all the eigenvalues ρ (multipliers) of the monodromy matrix $\mathbf{F}$ have a modulus of less than unity, and unstable if at least one multiplier has a modulus greater than unity.

Suppose that for a certain n-dimensional vector of the parameter $\mathbf{p}_0$ we know the monodromy matrix $\mathbf{F}_0 = \mathbf{F}(\mathbf{p}_0)$. We give the parameter vector an increment in the form $\mathbf{p} = \mathbf{p}_0 + \Delta\mathbf{p}$. As a consequence of this, the matrix $\mathbf{G}$, and consequently also $\mathbf{X}(t)$, obtained increments which lead to a change in the monodromy matrix $\mathbf{F}$. Expressions for the first and second derivatives of the monodromy matrix with respect to the parameters in the form of integrals over a period were obtained in [4, 5]

$$\begin{aligned}
\frac{\partial\mathbf{F}}{\partial p_k} &= \mathbf{F}_0\int_0^T \mathbf{H}_k(\tau)d\tau \\
\frac{\partial^2\mathbf{F}}{\partial p_i\partial p_j} &= \mathbf{F}_0\left[\int_0^T \mathbf{H}_{ij}(\tau)d\tau + \int_0^T \mathbf{H}_i(\tau)\left(\int_0^\tau \mathbf{H}_j(\zeta)d\zeta\right)d\tau + \int_0^T \mathbf{H}_j(\tau)\left(\int_0^\tau \mathbf{H}_i(\zeta)d\zeta\right)d\tau\right]
\end{aligned} \tag{2.3}$$

where

$$\mathbf{H}_k(\tau) = \mathbf{X}_0^{-1}(\tau)\frac{\partial\mathbf{G}}{\partial p_k}(\mathbf{p}_0, \tau)\mathbf{X}_0(\tau), \quad \mathbf{H}_{ij}(\tau) = \mathbf{X}_0^{-1}(\tau)\frac{\partial^2\mathbf{G}}{\partial p_i\partial p_j}(\mathbf{p}_0, \tau)\mathbf{X}_0(\tau); \quad i, j, k = 1, \ldots, n$$

The zero subscript denotes that the corresponding quantity is taken for $\mathbf{p} = \mathbf{p}_0$.

Note that to obtain the derivatives (2.3) it is only necessary to know the matricant $\mathbf{X}_0(t)$ and the derivatives of the matrix $\mathbf{G}$ with respect to the parameters, calculated for $\mathbf{p} = \mathbf{p}_0$. Using the derivatives (2.3), we can write increment of the monodromy matrix in the form

$$\mathbf{F}(\mathbf{p}_0 + \Delta\mathbf{p}) = \mathbf{F}_0 + \sum_{k=1}^{n}\frac{\partial\mathbf{F}}{\partial p_k}\Delta p_k + \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2\mathbf{F}}{\partial p_i\partial p_j}\Delta p_i\Delta p_j + \ldots \tag{2.4}$$

A knowledge of the derivatives of the monodromy matrix enables us to obtain the values of this matrix in the neighbourhood of the point $\mathbf{p}_0$ and, consequently, to estimate the behaviour of the multipliers (the eigenvalues of the monodromy matrix $\mathbf{F}$), responsible for the stability of system (2.1) when the parameters change.

3. REGIONS OF INSTABILITY

We will write system (1.5) in the form (2.1) with the matrix

$$\mathbf{G} = \begin{Vmatrix} 0 & [1 + 2\varepsilon\varsigma\varphi(\tau) + \varepsilon^2\varsigma\varphi^2(\tau)]^{-1} \\ -\omega^2 & -\beta\omega[1 + 2\varepsilon\varsigma\varphi(\tau) + \varepsilon^2\varsigma\varphi^2(\tau)]^{-1} \end{Vmatrix} \tag{3.1}$$

which depends explicitly on four parameters and the periodic function $\varphi(\tau)$. We will consider the behaviour of the multipliers in the neighbourhood of the point $\mathbf{p}_0 = (\omega, 0, 0)$ for an arbitrary value of the parameter $\varsigma \in (0, 1)$.

Substituting the values $\varepsilon = 0$ and $\beta = 0$ into relations (2.1) and (3.1), from Eqs (2.2) we obtain the matriciant and the matrix inverse to it

$$\mathbf{X}_0(t) = \begin{Vmatrix} \cos\omega t & \omega^{-1}\sin\omega t \\ -\omega\sin\omega t & \cos\omega t \end{Vmatrix}, \quad \mathbf{X}_0^{-1}(t) = \begin{Vmatrix} \cos\omega t & -\omega^{-1}\sin\omega t \\ \omega\sin\omega t & \cos\omega t \end{Vmatrix} \tag{3.2}$$

Hence, when $\varepsilon = 0$ and $\beta = 0$ the monodromy matrix has the form

$$\mathbf{F}_0 = \mathbf{X}_0(2\pi) \tag{3.3}$$

The eigenvalues of this matrix (the multipliers) are

$$\rho_{1,2} = \cos 2\pi\omega \pm i\sin 2\pi\omega \tag{3.4}$$

For all values of $\omega \neq k/2$ ($k = 1, 2, \ldots$) the multipliers are complex-conjugate quantities, and they lie on the unit circle (stability). For a small change in the parameters ω, ε, β in the neighbourhood of the point $\mathbf{p}_0 = (\omega, 0, 0)$, $\omega \neq k/2$ ($k = 1, 2, \ldots$), by virtue of continuity, the multipliers remain complex-conjugate quantities. Then, for the multipliers we have a quadratic equation of the form

$$\rho^2 + A\rho + B = 0 \tag{3.5}$$

The free term, according to Liouville's formula [2] using equality (3.1), is described by the expression

$$B = \exp\left(\int_0^{2\pi} \operatorname{tr}\mathbf{G}dt\right) = \exp(-2\pi\beta\omega(1 + o(\varepsilon^2))) \tag{3.6}$$

Since by Vieta's theorem, from relations (3.5) and (3.6) when $\beta > 0$ and sufficiently small ε we have

$$\rho_1\rho_2 = B < 1 \tag{3.7}$$

then for the complex-conjugate multipliers it follows from inequality (3.7) that $|\rho_{1,2}| < 1$. Hence, a small change in the parameters ω, β and ε, with $\beta > 0$, in the neighbourhood of the point $\mathbf{p}_0 = (\omega, 0, 0)$, $\omega \neq k/2$, shifts the multipliers inside the unit circle, which indicates asymptotic stability.

Consequently, instability (parametric resonance) can occur only in the neighbourhood of the points

$$\mathbf{p}_0 : \varepsilon = 0, \quad \beta = 0, \quad \omega = k/2, \quad k = 1, 2, \ldots \tag{3.8}$$

in which the multipliers are double.

To find the regions of parametric resonance we expand the monodromy matrix $\mathbf{F}$ in the neighbourhood of the points $\mathbf{p}_0$ in a Taylor series in the parameters ε, β and $\Delta\omega = \omega - k/2$

$$\mathbf{F}(\mathbf{p}) = \mathbf{F}(\mathbf{p}_0) + \frac{\partial\mathbf{F}}{\partial\varepsilon}\varepsilon + \frac{\partial\mathbf{F}}{\partial\beta}\beta + \frac{\partial\mathbf{F}}{\partial\omega}\Delta\omega + \ldots \tag{3.9}$$

From formulae (2.3) using relations (2.1) and (3.1)–(3.3) we can calculate the values of the derivatives $\partial\mathbf{F}/\partial\varepsilon$, $\partial\mathbf{F}/\partial\beta$ and $\partial\mathbf{F}/\partial\omega$, for $\mathbf{p} = \mathbf{p}_0$. As a result of expansion (3.9) we have, apart from terms of the first order of smallness.

$$\mathbf{F}(\mathbf{p}) = \cos\pi k \begin{Vmatrix} 1 + (\pi b_k k\varepsilon\varsigma - \pi\beta k)/2 & 4k^{-1}\pi\Delta\omega - \pi a_k\varepsilon\varsigma \\ -k\pi\Delta\omega - \pi a_k k^2\varepsilon\varsigma/4 & 1 - (\pi b_k k\varepsilon\varsigma + \pi\beta k)/2 \end{Vmatrix} \tag{3.10}$$

The constants

$$a_k = \frac{1}{\pi}\int_0^{2\pi} \varphi(\tau)\cos k\tau d\tau, \quad b_k = \frac{1}{\pi}\int_0^{2\pi} \varphi(\tau)\sin k\tau d\tau, \quad k = 1, 2, \ldots \tag{3.11}$$

are the Fourier coefficients of the function $\varphi(\tau)$.

For the matrix (3.10) we obtain approximate values of the multipliers

$$\rho_{1,2} = (-1)^k(1 - \pi\beta k/2) \pm \pi\sqrt{D}; \quad D = r_k^2 k^2\varepsilon^2\varsigma^2/4 - (2\Delta\omega)^2, \quad r_k = \sqrt{a_k^2 + b_k^2} \tag{3.12}$$

The system is unstable if at least one of the multipliers is larger in modulus than unity [2, 3]. When $\beta < 0$ this condition is satisfied and the system is unstable. But when $\beta \geq 0$ this condition is only satisfied when $\sqrt{D} > \beta k/2$. Hence, taking relations (3.12) into account, we obtain that the region of instability (parametric resonance) lies inside half the cone

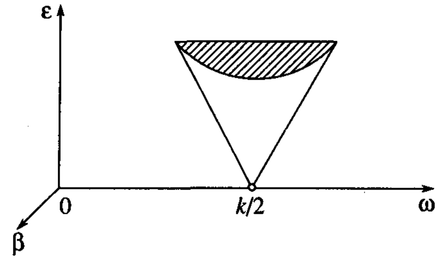

Fig. 2

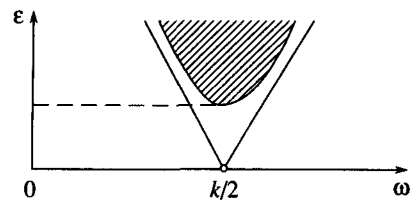

Fig. 3

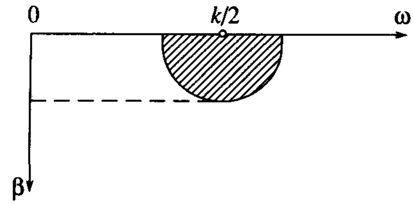

Fig. 4

$$4(2\omega/k-1)^2+\beta^2<r_k^2\varepsilon^2\varsigma^2, \quad \beta\geq 0 \tag{3.13}$$

connected with the half-space $\beta < 0$ (Fig. 2).

Hence, in particular, it follows that the kth region of parametric resonance depends only on the kth coefficients of the Fourier periodic excitation function. Note that formulae (3.13) are the first approximations for the regions of instability. When $a_k = 0$ and $b_k = 0$ the quantity $r_k = 0$, and the first-order approximations (3.13) degenerate into a straight line $\beta = 0$, $\omega = k/2$. In this case, in order to obtain the region of parametric resonance more accurately one must use higher-order approximations. This may also indicate that the corresponding region of parametric resonance is empty.

Putting $\beta = 0$ in inequality (3.13) we obtain the regions of parametric resonance when there is no damping

$$-\frac{r_k k\varsigma}{4}<\frac{\omega-k/2}{\varepsilon}<\frac{r_k k\varsigma}{4} \tag{3.14}$$

The section of the cone (3.13) by the plane $\beta = \text{const}$, $\beta \geq 0$ gives the regions of parametric resonance bounded by a hyperbola (Fig. 3), and its asymptotes are found from inequalities (3.14). When there is damping ($\beta > 0$) the minimum amplitude of the excitation of resonance, according to relation (3.13), is

$$\varepsilon_{\min} = \beta/r_k\varsigma \tag{3.15}$$

We will analyse the change in the regions of parametric resonance when the resonance number k increases. We know that, if the periodic function $\varphi(\tau)$ is continuous together with its sth order derivatives, then for the Fourier coefficients a_k and b_k we have the relations $a_k k^{s+1} \to 0$ and $b_k k^{s+1} \to 0$ when $k \to \infty$. Hence, for continuously differentiable functions, the quantities kr_k approach zero when $k \to \infty$. This means that the cone (3.13) shrinks as k increases. Hence it also follows that for fixed β the minimum amplitude of excitation of resonance (3.15) increases without limit as k increases. This is explained by the fact that it is easier to observe resonance at low values of $k = 1, 2$, whereas to achieve resonance at higher values of k larger excitation amplitudes are required.

The section of the region (3.13) by the plane $\varepsilon = \text{const}$ is half an ellipse with semiaxes $|\omega - k/2| = r_k k\varepsilon\varsigma/4$ and $\beta = r_k\varepsilon\varsigma$ (Fig. 4). Note that as the damping coefficient β increases the width of the region of parametric resonance shrinks with respect to the frequency ω and when $\beta > r_k\varepsilon\varsigma$ it disappears.

It follows from relation (3.13) and the above analysis that the regions of parametric resonance broaden as the parameter $\varsigma \in (0, 1)$ increases. According to relations (1.4) this means that, when the moment of inertia of the masses increases and the moment of inertia of the disc remains unchanged, the regions of parametric resonance broaden.

Using relations (1.4) and (3.8) we find that parametric resonance occurs in the neighbourhood of the dimensional critical frequencies

$$\Omega_{cr} = \frac{2\Omega_0}{k}, \quad k = 1, 2, \ldots; \quad \Omega_0 = \sqrt{\frac{c}{J_0 + 2mr_0^2}} \tag{3.16}$$

Note that Ω_0 is the natural frequency of torsional oscillations of a disc with two fixed masses m, situated at a distance r_0 from the axis of rotation. Using relations (1.4) and (3.16) the regions of parametric resonance (3.13) can be written in dimensional variables.

As a numerical example we will consider the periodic function $\varphi(\tau) = \cos\tau$ and the parameter $\varsigma = 1/2$. In this case, for the first resonance region $k = 1$ we calculate $a_1 = r_1 = 1$ and $b_1 = 0$ using expressions (3.11). Hence, according to relations (1.4) and (3.13) we obtain the following explicit expression for the first region of parametric resonance in dimensional variables

$$4\left(\frac{\Omega}{2\Omega_0} - 1\right)^2 + \frac{\gamma^2}{(J_0 + 2mr_0^2)c} < \frac{a^2}{4r_0^2} \tag{3.17}$$

Note that other regions of parametric resonance are degenerate, since $a_k = b_k = r_k = 0$ ($k = 2, 3, 4, \ldots$).

We will consider, as another example, the continuously differentiable 2π-periodic function

$$\varphi(\tau) = \begin{cases} \tau(\pi - \tau), & 0 \le \tau \le \pi \\ (\pi - \tau)(2\pi - \tau), & \pi < \tau \le 2\pi \end{cases} \tag{3.18}$$

consisting of parabolas of positive and negative curvature. We have for the Fourier coefficients of this function

$$a_k = 0, \quad b_k = r_k = \frac{8}{\pi k^3}, \quad k = 1, 3, 5, \ldots; \quad a_k = b_k = r_k = 0, \quad k = 2, 4, 6, \ldots \tag{3.19}$$

Hence, by formula (3.13), for the odd resonances we obtain the relation

$$4\left(\frac{2\omega}{k} - 1\right)^2 + \beta^2 < \frac{64\varepsilon^2\varsigma^2}{\pi^2 k^6}, \quad k = 1, 3, 5, \ldots \tag{3.20}$$

which shows how rapidly the regions of parametric resonance shrink as the number k increases, while the even regions of parametric resonance are degenerate.

The above method of analysing the regions of parametric resonance, which uses the derivatives of the monodromy matrix with respect to the parameters, is a simpler and more effective method compared with methods based on searching for periodic solutions at the boundaries of the regions of stability [3]. Moreover, it gives relations for the regions of stability and not just for the boundaries of stability, and gives the multipliers which describe the nature of unstable motion.

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